PERIDYNAMIC THEORY

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Introduction

Peridynamics introduced by Silling [1] is one of the nonlocal methods which reformulate the classical continuum mechanics by substituting the governing partial differential equations with integral equations. Therefore, the peridynamic equation of motion holds true anywhere in a rigid body, despite the presence of cracks and other discontinuities. Furthermore, material damage is an inherent feature and thus a key component of the peridynamic constitutive laws. These attributes permit modeling of fracture initiation and propagation, with arbitrary paths, i.e., without the need for a crack growth treatment.

Bond-Based Peridynamics (BB-PD)

Peridynamics assumes that any material point, $\mathbf{x}$, which belongs to the spatial region, R, interacts with any arbitrary material point, $\mathbf{x}'$, within a finite distance of δ. The interaction between the material points is defined by a "bond" relationship between material points. In the first introduced peridynamics method, so called the "bond-based peridynamics", the interaction between material points is limited to a pair-wise force function, $\mathbf{f}$, defined by Eq. (1), as illustrated in Fig. 1a:

$$\mathbf{f} = \mathbf{f}\big(\mathbf{x}, \mathbf{x}', \mathbf{u}(\mathbf{x}, t), \mathbf{u}(\mathbf{x}', t)\big) \qquad |\mathbf{x}' - \mathbf{x}| \leq \delta \tag{1}$$

where $\mathbf{u}$ denotes the displacement field, and t is time. Therefore, the acceleration of material point, $\mathbf{x}$, is obtained from a summation of all bond forces as shown in Eq. (2)

$$\rho\ddot{\mathbf{u}}(\mathbf{x}, t) = \int_{H_\mathbf{x}} \mathbf{f}\big(\mathbf{x}, \mathbf{x}', \mathbf{u}(\mathbf{x}, t), \mathbf{u}(\mathbf{x}', t)\big) dV_{\mathbf{x}'} + \mathbf{b}(\mathbf{x}, t) \tag{2}$$

where ρ is mass density in the reference configuration, $dV_{\mathbf{x}'}$ is an infinitesimal volume associated with material point $\mathbf{x}'$, $\mathbf{b}$ is the external body force density field, and $H_\mathbf{x}$ is the domain of neighboring material points and is referred to as "horizon".

In peridynamic literature, the relative position of two points $\mathbf{x}$ and $\mathbf{x}'$ in the reference configuration is denoted by $\boldsymbol{\xi}$

$$\boldsymbol{\xi} = \mathbf{x}' - \mathbf{x} \tag{3}$$

and the relative displacement is denoted by $\boldsymbol{\eta}$

$$\boldsymbol{\eta} = \mathbf{u}'(\mathbf{x}', t) - \mathbf{u}(\mathbf{x}, t) \tag{4}$$

Constitutive modeling: For a homogeneous material which does not result in time-dependent deformations, the magnitude of pair-wise force, $\mathbf{f}$, is obtained as a function of stretch, s, in the form of $f = f(s)$, where the stretch, s, is defined by

$$s = \frac{|\boldsymbol{\xi}+\boldsymbol{\eta}| - |\boldsymbol{\xi}|}{|\boldsymbol{\xi}|} \tag{5}$$

In Eq. (5), $\boldsymbol{\xi} + \boldsymbol{\eta}$ represents the relative position in the deformed configuration, $|\boldsymbol{\xi}|$ is the undeformed bond length, and $|\boldsymbol{\xi} + \boldsymbol{\eta}|$ is the deformed bond length. Therefore, the pair-wise force function takes the form of

$$\mathbf{f} = \omega(|\boldsymbol{\xi}|) f(s) \frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{|\boldsymbol{\xi}+\boldsymbol{\eta}|} \tag{6}$$

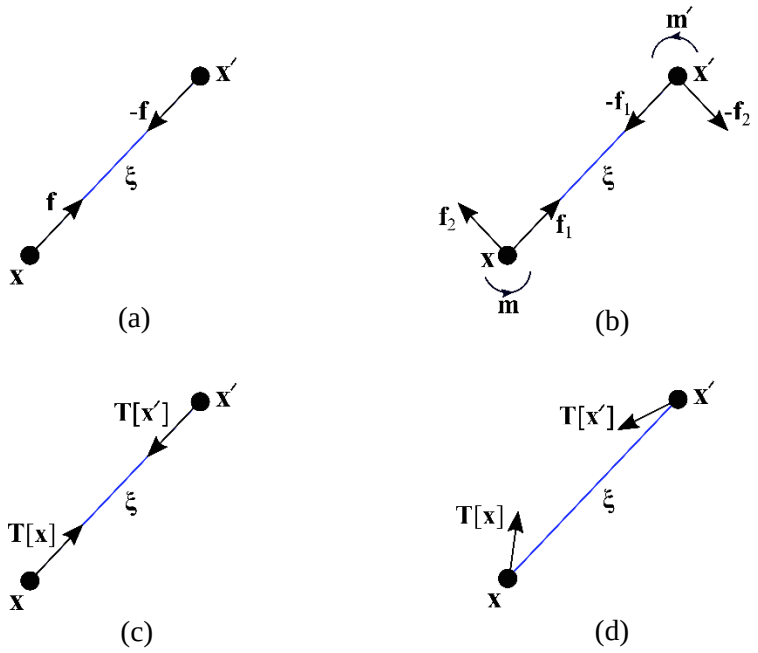

Figure 1. Schematic of force components between two material particles in the (a) bond-based, (b) micropolar, (c) ordinary state-based and (d) non-ordinary state-based peridynamics.

where $\omega(|\boldsymbol{\xi}|)$ is a radially symmetric weight function and $\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{|\boldsymbol{\xi}+\boldsymbol{\eta}|}$ denotes the unit vector of current relative position of two points. It is noteworthy that the pair-wise force function, $f(s)$, contains the material constitutive information. For example, for elastic materials, f, is a linear function of s (i.e., $f = cs$), where the stiffness c depends on both material and the geometrical property of the material. c, so called "material micromodulus", is generally obtained directly by an empirical study. Silling and Askari [2] obtained the material micromodulus by comparing the peridynamic micropotential with the strain energy density in the classical theory of elasticity for the same material, noting that, the pair-wise force is defined by a nonlinear function of the stretch, s, for materials with nonlinearity and damage parameters.

Damage in the peridynamic theory is associated with bond breakage between material points. A bond can break if its stretch, s, exceeds a critical stretch, s_0. The critical stretch can be obtained experimentally. Furthermore, Silling and Askari [2] represented a formulation to calculate the critical stretch by comparing the fracture energy of the material, G_0, with the total energy required to generate a fracture surface within the horizon of a material point. The simplest way to introduce failure into peridynamic formulation is to modify the weight function, $\omega(|\boldsymbol{\xi}|)$, introducing additional term to capture the bond breakage as follows:

$$\bar{\omega}(|\boldsymbol{\xi}|) = \begin{cases} \omega(|\boldsymbol{\xi}|) & s < s_0 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Horizon size: A constitutive parameter, called 'the horizon', in the peridynamic theory represents the extent (e.g., the radius of influence) of nonlocal interactions in a continuum model. The peridynamic horizon is the region around a material point and usually takes as a circular domain centered at the point **x**, with a radius of δ, in a two-dimensional problem. In a similar manner, the horizon becomes a sphere in three-dimensional problems. Nonlocality is a key component of peridynamic formulations because the

spatial derivative term is replaced with an integral term to overcome shortcomings of the classical mechanics in modeling discontinuities.

In certain problems where the physical interactions between points are visible, the horizon size can be correlated to an intrinsic material length-scale [3–5]. However, the actual length scale is undetectable at the macroscale [4, 5]. In such cases, the horizon size is mostly expressed as a function of discretization (or grid) size using a comprehensive convergence study to predict the results of experimental data. For macroscopic modeling, it is reported in several studies where the selection of the peridynamic horizon size is approximately three times the grid size, the results is obtained with reasonable stability [2, 3, 6–9]. Smaller horizon size is known to result in undesirable grid effects [8]. For larger horizons, it is reported that excessive wave dispersion could occur for dynamic solutions [8]. Furthermore, a larger horizon size lowers efficiency as the total number of interactions between material points increases. It will also lead to increased computation time and a higher degree of difficulty for implementation.

It is important to note that a relatively large horizon at the macroscale (e.g., $1\ mm$) does not indicate that there is a physical connection between material points spaced at such distance. It simply indicates that the material response such as linear/nonlinear deformation or damage evolution is well captured by the peridynamic model with such horizon size [8].

It is also important to recognize that the horizon size does not have to be constant over the domain. In certain problems involving adaptive mesh refinement around a stress concentration zone [3, 5] or multiscale modeling [10], spatial distribution of points with variable horizon size is necessary for computational efficiency. However, when the horizon size varies between points, the bond forces are no longer pairwise which could result in spurious wave reflections in solutions. Refined peridynamics [3,5] and dual horizon concept [11] are been studied by researchers in an effort to deal with spurious wave reflection issues and ghost force problems between points.

Fixed Poisson's ratio in bond-based peridynamics: In the bond-based peridynamics, two points interact utilizing the central pair-wise bond force concept. The assumption results in a fixed Poisson's ratio for all bond-based peridynamic materials. Poisson's ratio can be obtained by comparing the stress tensor under small homogeneous deformation for linear micro-elastic bond-based peridynamic body with the classic stress tensor under the same deformation field [12]. For a linear micro-elastic bond-based peridynamic material undergoing small homogeneous deformation in three-dimensional, the Poisson's ratio is obtained as 1/4 [12]. In two-dimensional bond-based peridynamic problems, the Poisson's ratio is 1/3 and 1/4 for plane stress and plane strain problems, respectively [2].

Micropolar Peridynamics

A generalized peridynamic formulation is developed by Grestle et al. [13] to address the deficiency of the bond-based peridynamics in modeling material behavior with varying Poisson's ratio. The so called "micropolar peridynamics" allows pair-wise moments to act between the material points in addition to pairwise force, as illustrated in Fig. 1b. Therefore, in addition to Eq. (2), the equation of motion is expressed in terms of moments as presented in Eq. (9), where $\boldsymbol{\theta}$ denotes the material particle rotation; I is the mass moment of inertia, $\mathbf{m}$ is the pair-wise peridynamic moments; and $\mathbf{m_b}$ is the externally applied moment vector per unit volume.

$$\rho\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{H_\mathbf{x}} \mathbf{f}\big(\mathbf{x},\mathbf{x}',\mathbf{u}(\mathbf{x},t),\mathbf{u}(\mathbf{x}',t),\boldsymbol{\theta}(\mathbf{x},t),\boldsymbol{\theta}(\mathbf{x}',t)\big)dV_{\mathbf{x}'} + \mathbf{b}(\mathbf{x},t) \tag{8}$$

$$I\ddot{\boldsymbol{\theta}}(\mathbf{x},t)=\int_{H_{\mathbf{x}}}\mathbf{m}\big(\mathbf{x},\mathbf{x}',\mathbf{u}(\mathbf{x},t),\mathbf{u}(\mathbf{x}',t),\boldsymbol{\theta}(\mathbf{x},t),\boldsymbol{\theta}(\mathbf{x}',t)\big)dV_{\mathbf{x}'}+\mathbf{m}_{\boldsymbol{b}}(\mathbf{x},t)(\tag{9}$$

Generally, the micropolar peridynamics [13] assumes that the bond relationship between two material points interact as a beam element, rather than a truss element used in the bond-based peridynamics [12]. In the case of quasi-static motion, the accelerations are zero, so that for each point,

$$\int_{H_{\mathbf{x}}}\mathbf{f}\big(\mathbf{x},\mathbf{x}',\mathbf{u}(\mathbf{x},t),\mathbf{u}(\mathbf{x}',t),\boldsymbol{\theta}(\mathbf{x},t),\boldsymbol{\theta}(\mathbf{x}',t)\big)dV_{\mathbf{x}'}+\mathbf{b}(\mathbf{x},t)=\mathbf{0} \tag{10}$$

$$\int_{H_{\mathbf{x}}}\mathbf{m}\big(\mathbf{x},\mathbf{x}',\mathbf{u}(\mathbf{x},t),\mathbf{u}(\mathbf{x}',t),\boldsymbol{\theta}(\mathbf{x},t),\boldsymbol{\theta}(\mathbf{x}',t)\big)dV_{\mathbf{x}'}+\mathbf{m}_{\boldsymbol{b}}(\mathbf{x},t)=\mathbf{0} \tag{11}$$

State-based Peridynamics (SB-PD)

Introducing rotational degrees of freedom into a bond-based peridynamic model, the micropolar peridynamics offers an important improvement over the original theory in modeling material behavior with varying Poisson's ratio. However, it remains difficult to represent all aspects of material response, particularly when a collective response among material point bonds was involved (e.g., a volume change). To avoid this type of restriction, a generalized peridynamic formulation, so-called state-based peridynamics was introduced. The state-based peridynamics method allows interaction between bonds [14]. Therefore, the collective deformation of the bonds around a material point defines the response of the point [14].

In the state-based peridynamic formulation, the equation of motion in Eq. (2) is replaced by:

$$\rho\ddot{\mathbf{u}}(\mathbf{x},t)=\int_{H_{\mathbf{x}}}(\mathbf{T}[\mathbf{x},t]\langle\mathbf{x}'-\mathbf{x}\rangle-\mathbf{T}[\mathbf{x}',t]\langle\mathbf{x}-\mathbf{x}'\rangle)dV_{\mathbf{x}'}+\mathbf{b}(\mathbf{x},t) \tag{12}$$

where $\mathbf{T}[\mathbf{x},t]$ is the force vector state field applied on material point $\mathbf{x}$ at time t. The angle bracket, $\langle\ \ \rangle$, is used to indicate that the state operates on the vector in $\langle\ \ \rangle$. To determine the force vector state, $\mathbf{T}$, two formulations were developed: ordinary method and non-ordinary method.

Ordinary formulation: In the ordinary state-based peridynamics, the force density vectors having unequal magnitudes while being parallel as shown in Fig. 1c. Thus, they can be defined in the form

$$\mathbf{T}[\mathbf{x},t]\langle\mathbf{x}'-\mathbf{x}\rangle=\frac{1}{2}As\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{|\boldsymbol{\xi}+\boldsymbol{\eta}|} \tag{13}$$

$$\mathbf{T}[\mathbf{x}',t]\langle\mathbf{x}-\mathbf{x}'\rangle=-\frac{1}{2}Bs\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{|\boldsymbol{\xi}+\boldsymbol{\eta}|} \tag{14}$$

where A and B are the auxiliary parameters that are dependent on engineering material constants, deformation field, and the horizon. The selection force density vectors in this form referred by Silling et al. [14] as "ordinary state-based" peridynamics (OSB-PD).

Non-ordinary formulation: The main goal of NOSB-PD was to allow interactions among the bonds [15]. Therefore, the collective deformation of the bonds around a material point defines the response of the point [15,16]. In NOSB-PD, each bond between two material points is capable of carrying loads in all directions (See Fig. 1d), which allows characterizing materials with any thermodynamically-admissible Poisson's ratio [14–16]. Furthermore, the NOSB-PD is capable of representing genuine material behaviors, such as a volume or shear angle change. Furthermore, the force state is represented by the classical stress and strain tensors, which enable the use of constitutive and damage models from the classical mechanics theory.

In the state-based peridynamics, the relative position of two material points is defined as the position state, $\mathbf{X}\langle \mathbf{x}' - \mathbf{x} \rangle = \boldsymbol{\xi} = \mathbf{x}' - \mathbf{x}$. The deformation state, $\mathbf{Y}$, is the position state in the deformed configuration and expressed as:

$$\mathbf{Y}\langle \mathbf{x}' - \mathbf{x} \rangle = \boldsymbol{\xi} + \boldsymbol{\eta} = \mathbf{y}' - \mathbf{y} \tag{15}$$

The deformation state, $\mathbf{Y}$, represents the basic kinematic quantity used for a constitutive model in the state-based peridynamic method. Therefore, for an elastic material, the strain energy density, W, is given by a function of the deformation state, instead of deformation gradient tensor, $\mathbf{F} = \partial \mathbf{y}/\partial \mathbf{x}$, used in the classical continuum mechanics. In order to employ a constitutive model from the classical solid mechanics in the NOSBPD, a deformation gradient tensor must be described in forms of peridynamic states. An approximation of the nonlocal deformation gradient tensor was proposed by Silling et al. [14] as:

$$\mathbf{F}(\mathbf{x}) = \left[\int_{H_\mathbf{x}} \omega(|\boldsymbol{\xi}|)[(\mathbf{y}' - \mathbf{y}) \otimes (\mathbf{x}' - \mathbf{x})] dV_{\mathbf{x}'}\right] . \mathbf{K}^{-1} \tag{16}$$

where $\otimes$ denotes dyadic product and $\mathbf{K}$ is a shape tensor, expressed as follows at point $\mathbf{x}$.

$$\mathbf{K}(\mathbf{x}) = \int_{H_\mathbf{x}} \omega(|\boldsymbol{\xi}|)[(\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x})] dV_{\mathbf{x}'} \tag{17}$$

The force state $\mathbf{T}$ is defined in terms of changes in strain energy density, $W(\mathbf{Y})$, for any infinitesimal change in the deformation state, $d\mathbf{Y}$, and formulated as follows [15]:

$$\mathbf{T}[\mathbf{x}, t]\langle \mathbf{x}' - \mathbf{x} \rangle = \omega(|\boldsymbol{\xi}|)\boldsymbol{\sigma} . \mathbf{K}^{-1} . (\mathbf{x}' - \mathbf{x}) \tag{18}$$

$\boldsymbol{\sigma} = \partial W/\partial \mathbf{F}$ is the first Piola-Kirchhoff stress. For an isotropic elastic material with small strain tensor of $\boldsymbol{\varepsilon} = (\mathbf{F} + \mathbf{F}^T)/2 - \mathbf{I}$, the stress is obtained through $\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}$, where $\mathbf{C}$ is the isotropic elastic moduli matrix and $\mathbf{I}$ is the identity matrix.

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